Journal of Applied Mechanics and Technical Physics, Vol. 45, No. 3, pp. 375–378, 2004

FORMULA FOR THE FLOW RESISTANCE FACTOR IN A PIPE WITH A SUDDEN EXPANSION AT SMALL REYNOLDS NUMBERS

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UDC 621.541

A formula for the flow resistance factors in a pipe with a sudden expansion of the cross section at Reynolds numbers of 0.2 *to* 10 *is obtained by numerical solution of the complete Navier–Stokes equations for incompressible fluids. The flow resistance factors obtained using the derived formula are compared to those found by numerical solution of the Navier–Stokes equations.*

Key words: *cylindrical pipe, sudden expansion, small Reynolds numbers, resistance.*

For calculations of the local resistance factors of a cylindrical channel with a sudden expansion at small Reynolds numbers ($\mathrm{Re} \leqslant 10$) and a small expansion ratio ($D_2/D_1 \leqslant 3$), Idel'chik recommended [1] the formula $\xi = 30/\mathrm{Re}$ irrespective of the diameter ratio D_2/D_1 and the channel lengths before and after the expansion.

We consider the local resistance of a channel in the form of two cylindrical pipes of different diameters (Fig. 1). At the entrance to the pipe of diameter D_1, there is Poiseuille flow with a parabolic velocity distribution over the cross section [2] $U_1 = 2(1 - 4\bar{r}^2)$, where $\bar{r} = r/D_1$ is the dimensionless current radius, and $U_1 = u_1/w_0$ (U_1 is the current velocity and w_0 is the mean flow rate). At a large distance from the place of expansion Poiseuille flow occurs that corresponds to the larger cross section of the pipe. Let $D_2/D_1 = b$; then in the pipe of diameter D_2, Poiseuille flow with the velocity distribution $U_2 = 2(b^2 - 4\bar{r}^2)/b^4$ is established.

We designate the Reynolds number $\mathrm{Re} = w_0 D_1/\nu$, where ν is the liquid viscosity. Then, the dimensionless pressure gradients at the entrance and exit are equal to $-32/\mathrm{Re}$ and $-32/(\mathrm{Re}\, b^4)$, respectively.

In this formulation, the flow was studied numerically invoking the Navier–Stokes equations for incompressible liquids. The total-pressure loss and the resistance factor were determined. The mathematical formulation of the problem and the method of solution are described in [3, 4]. Here we analyze some calculation results, whose analysis indicates the applicability of the formula derived below. Cross sections 1 and 2 (see Fig. 1) located at distances l_1 and l_2 upstream and downstream, respectively, from the cross-section change were chosen. In these cross sections, the static pressure p and the velocity head $\zeta U^2/2$ were determined numerically and used to find the resistance factor (ζ is the dimensionless density). For Reynolds numbers $10 < \mathrm{Re} < 200$ and various D_2/D_1, the obtained values of the resistance factor are tabulated in [3, 4].

Figures 2 and 3 give calculated distributions of the dimensionless pressure gradient along the channel length along the streamline located near the symmetry axis. From these figures, it is possible to examine the pressure variation. From the entrance cross section, in which there is a sudden expansion of the channel (denoted by a prime), the pressure decreases under Poiseuille's law. No perturbations propagate upstream, and the Poiseuille flow is conserved with good accuracy until the channel expansion. Immediately behind the expansion, the pressure

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Computer Center, Russian Academy of Sciences, Moscow 119991. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 45, No. 3, pp. 81–84, May–June, 2004. Original article submitted December 24, 2002; revision submitted June 16, 2003.

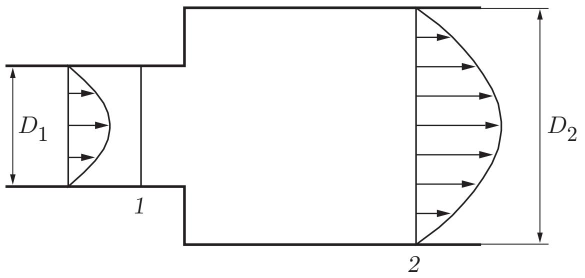

Fig. 1

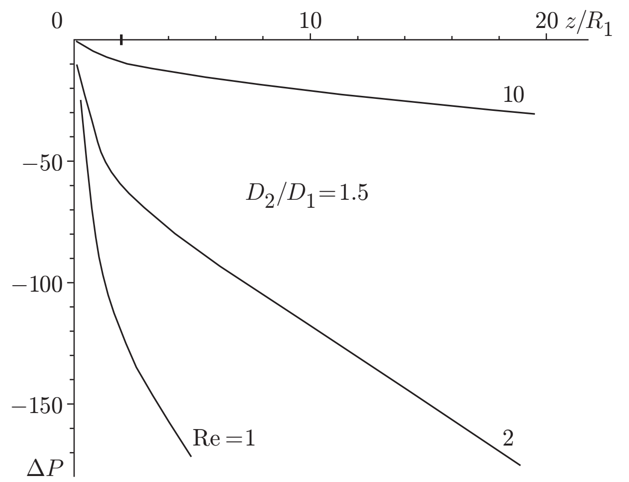

Fig. 2

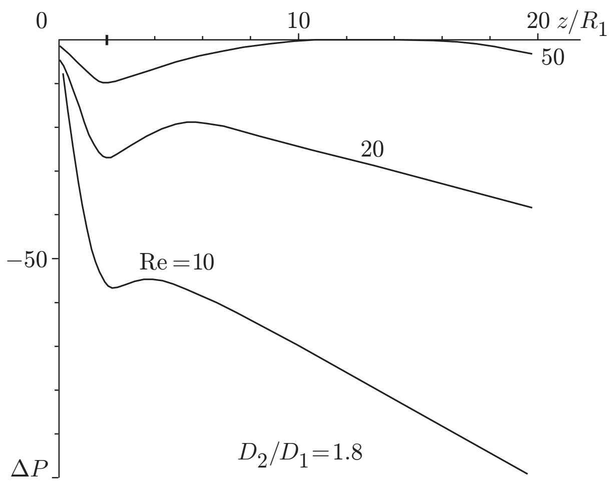

Fig. 3

increases, reaches the maximum value, and then begins to decrease, entering the Poiseuille distribution corresponding to the pipe exit diameter. This effect is differently manifested, depending on the value of Re. With increase in Re numbers, the transitional region is enlarged and the maximum value of p increases, whereas at small Re numbers, the transitional region is insignificant. The variation in the velocity head is shown in Fig. 4. At a rather large distance downstream, the velocity head losses are determined only by the value of D_2/D_1 and do not depend on Re. However, the values of Re influence the location and pattern of this transition. From Fig. 3, it is evident that with increase in Re, the region of the transition is enlarged. However, at small Re numbers, this region is almost absent.

These facts underlie the derivation of the formula for the resistance factor in the presence of an expansion. It is assumed that the transitional zone can be ignored and that the Poiseuille flow in the pipe of the smaller diameter is followed by the Poiseuille flow corresponding to the larger pipe diameter.

Let us derive this formula. We calculate the values of $\zeta U_1^2/2$ at the entrance, where U_1^2 is the mean squared dimensionless velocity in the smaller diameter pipe. The value of U_1^2 is calculated by the formula

$$U_1^2 = \frac{4}{\pi D_1^2} \int_0^{D_1/2} 2\pi r u_1^2 \, dr = \int_0^{1/2} 4(1 - 4\bar{r}^2)^2 8\bar{r} \, d\bar{r} = \frac{4}{3}.$$

We next calculate the value of U_2^2 (at the exit) using the same formula, where U_2^2 is mean squared dimensionless velocity in the larger diameter pipe:

$$U_2^2 = \frac{4}{\pi D_1^2 b^2} \int_0^{bD_1/2} 2\pi r u_2^2 \, dr = \frac{4}{b^2} \int_0^{b/2} \frac{8}{b^8} (b^2 - 4\bar{r}^2)^2 \bar{r} \, d\bar{r} = \frac{4}{3b^4}.$$

The pressure gradient from the entrance cross-section to the place of expansion is determined from the formula

$$\Delta p_1 = -32 l_1 / \mathrm{Re},$$

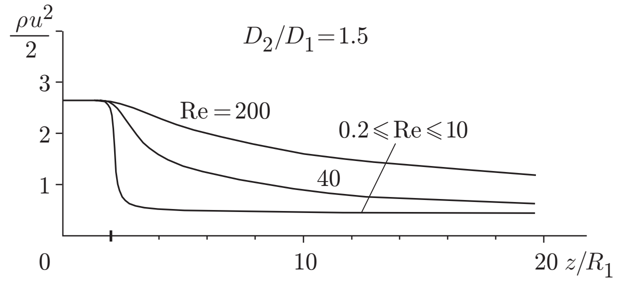

Fig. 4

TABLE 1

$D_2/D_1 = 1.2$

Re	Values of ξ for			
	$l_2 = 2$	$l_2 = 4$	$l_2 = 6$	$l_2 = 8$
0.2	427.81/472.00	659.44/703.48	775.26/934.96	1006.89/1166.44
1	85.35/94.81	131.68/141.11	154.85/187.41	201.18/233.70
2	42.55/47.67	65.72/70.81	77.30/93.96	100.47/117.11
10	8.36/9.95	13.00/14.58	15.32/19.21	19.96/23.84

TABLE 2

$D_2/D_1 = 2.0$

Re	Values of ξ for			
	$l_2 = 2$	$l_2 = 4$	$l_2 = 6$	$l_2 = 8$
0.2	259.08/270.94	289.13/300.94	304.16/330.94	334.21/360.94
1	51.41/54.94	57.42/60.94	60.42/66.94	66.43/72.94
2	25.46/27.94	28.47/30.94	29.97/33.94	32.98/36.94
10	4.88/6.34	5.48/6.94	5.78/7.54	6.38/8.14

TABLE 3

$D_2/D_1 = 3.0$

Re	Values of ξ for			
	$l_2 = 2$	$l_2 = 4$	$l_2 = 6$	$l_2 = 8$
0.2	242.61/246.91	248.55/252.84	251.52/258.77	257.46/264.69
1	48.10/50.17	49.29/51.36	49.88/52.54	51.07/53.73
2	23.81/25.58	24.40/26.17	24.70/26.77	25.29/27.36
10	4.62/5.91	4.74/6.02	4.80/6.14	4.92/6.26

where l_1 is the dimensionless length of the pipe of diameter D_1 and z is the dimensionless coordinate. It is assumed that the pressure gradient in the pipe of diameter D_1 is constant: $dp/dz = -32/\mathrm{Re}$. Similarly, assuming that the pressure gradient in the wide pipe is constant and equal to $dp/dz = -32/(\mathrm{Re}\, b^4)$, we obtain

$$p_2 = -\frac{32}{\mathrm{Re}}\, l_1 - \frac{32}{\mathrm{Re}\, b^4}\, l_2,$$

where l_2 is the dimensionless length of the pipe of diameter D_2.

The resistance factor is calculated as follows:

$$\xi = \frac{2}{\zeta U_1^2}\Big[\Big(p_1 + \frac{\zeta U_1^2}{2}\Big) - \Big(p_2 + \frac{\zeta U_2^2}{2}\Big)\Big] = \frac{3}{2}\Big(\frac{32}{\mathrm{Re}}\, l_1 + \frac{32}{\mathrm{Re}\, b^4}\, l_2 + \frac{2}{3} - \frac{2}{3b^4}\Big) = \frac{48}{\mathrm{Re}}\Big(l_1 + \frac{l_2}{b^4}\Big) + 1 - \frac{1}{b^4}. \qquad (1)$$

In the derivation of this formula, the diameter of the entrance pipe is set equal to unity; therefore, l_1 and l_2 should be normalized by its value.

Tables 1–3 give the resistance factors for Reynolds numbers Re = 0.2–10 and various values of D_2/D_1 [the numerator contains the resistance factors obtained by numerical solution of the Navier–Stokes equations, and the denominator contains the resistance factors obtained using formula (1)]. Here $l_1 = 1$ and $l_2 = 2, 4, 6, 8$.

As can be seen from the tables, in the given range of Re and l_2 for $D_2/D_1 \leqslant 3$, there is good agreement between the values of ξ obtained by integration of the Navier–Stokes equations and those calculated by formula (1). The maximum difference of 27% is observed for Re = 10. A comparison for Re = 10 shows that the resistance factors obtained by formula (1) differ markedly form the value of $\xi = 3$ given in [1]. It is obvious that the factor ξ depends on the Re, D_2/D_1, l_1, and l_2. For Re = 10, calculations using the proposed formula for $l_1 = 1$, $l_2 = 2$, and $D_2/D_1 = 1.2$, 2, and 3 give values $\xi = 9.95$, 6.34, and 5.91. For $l_1 = 0.5$ and $l_2 = 2$, the corresponding values of ξ are equal to 7.55, 3.94, and 3.5. These values are closer to the value $\xi = 3$ from [1]. However, an increase in l_2 leads to larger differences.

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